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# Intersections of quadrics with spheres

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**Abstract.** In Euclidean space $\mathbb{R}^{n+2}$ we study the intersections of central quadrics with spheres where we consider the intersections as hypersurfaces in $\mathbb{S}^{n+1}(1)$. It is our aim to characterize such intersections within the class of all hypersurfaces in $\mathbb{S}^{n+1}(1)$ with Weingarten operator of maximal rank. The methods we use are similar to methods from affine hypersurface theory.

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## 1. Introduction

A hypersurface immersion in a space form is called non-degenerate if the second fundamental form defines a semi-Riemannian metric. The symmetric difference tensor field $C := \nabla^1 - \nabla^2$, where $\nabla^1$ and $\nabla^2$ are the Levi-Civita connections of the first fundamental form (induced metric) and the second fundamental form, resp., has interesting geometric properties. In Blaschke's affine hypersurface theory, the analogous equation $C = 0$ characterizes quadrics, while in the general relative geometry quadrics are characterized by the equation $\widetilde{C} = 0$, [5], [6], where $\widetilde{C}$ is the traceless part of the difference tensor field $C$. In this paper we study intersections of hyperquadrics and hyperspheres in Euclidean space $\mathbb{R}^{n+2}$; we consider such intersections as hypersurfaces in $\mathbb{S}^{n+1}(1)$ and assume the hypersurfaces to be non-degenerate. Do such intersections satisfy the equation $\widetilde{C} = 0$ as regular immersions in spheres? We observe that if a quadric and a sphere are centered at the same point, then the intersection has this property; otherwise, the necessary condition for such intersection to satisfy the condition $\widetilde{C} = 0$ is that the immersion has to be totally umbilical in $\mathbb{S}^{n+1}$. The next question is to know whether a (hyper)surface of a sphere $\mathbb{S}^{n+1}$ satisfying the condition $\widetilde{C} = 0$ is contained in an open part of some quadric $\mathbb{Q}^{n+1}$ ($\neq \mathbb{S}^{n+1}$) of $\mathbb{R}^{n+2}$. We investigate

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this situation for $n \geq 2$ and in detail for dimension 2; we use results from [4] where non-isoparametric regular immersions without umbilics in $\mathbb{S}^3(1)$ are completely classified. We characterize surface immersions in $\mathbb{S}^3(1)$ satisfying $\widetilde{C} = 0$ in terms of Euclidean quadrics.

## 2. Notations and basic facts

Let $x\colon (M^n, \mathrm{I}) \longrightarrow \mathbb{S}^{n+1}(1) \subset \mathbb{R}^{n+2}$ be an isometric immersion of a connected and orientable, $n$-dimensional $C^\infty$-manifold $M^n$ into the Euclidean sphere

$$\mathbb{S}^{n+1}(1) = \{y = (y_1, \cdots, y_{n+2}) \in \mathbb{R}^{n+2} \colon \|y\|^2 =< y, y >= \sum_{i=1}^{n+2} y_i^2 = 1\}.$$

Denote by $N$ a unit vector field on $\mathbb{S}^{n+1}(1)$ normal to $M^n$, by $<,>$ the canonical inner product of the Euclidean structure and by $\bar{\nabla}$ the associated Levi-Civita connection. Assume that the immersion $x$ is non-degenerate (regular), i.e the second fundamental form is of maximal rank on $M^n$. In this case the spherical Gauß map $N$ defines a regular immersion $x^*\colon M^n \longrightarrow \mathbb{S}^{n+1}(1)$ called the polarized hypersurface of $x$; the pair $(x, x^*)$ of hypersurfaces in $\mathbb{S}^{n+1}(1)$ is also called a polar pair [1]; that means

$$< x, dx^* >= 0, \quad < x^*, dx >= 0, \quad < x, x^* >= 0. \tag{2.1}$$

The immersion $x$ and its polarized hypersurface $x^*$ have the same second fundamental form $\mathrm{II}$. The first fundamental form $\mathrm{I}^*$ of $x^*$ coincides with the third fundamental $\mathrm{III}$ form of $x$ and vice-versa; $x$ is umbilical if and only if $x^*$ is umbilical. Denote by $S$ the shape operator of $x$. $S^* = S^{^{-1}}$ is the shape operator of $x^*$. The fundamental equations (Gauß and Weingarten equations) for the immersions $x$ and $x^*$ are given by:

$$\begin{aligned}
\bar{\nabla}_u dx(v) &= dx(\nabla^1_u v) + \mathrm{II}(u,v)x^* - \mathrm{I}(u,v)x; & (2.2)\\
\bar{\nabla}_u dx^*(v) &= dx^*(\nabla^3_u v) + \mathrm{II}(u,v)x - \mathrm{III}(u,v)x^*; & (2.3)\\
dx^*(v) &= -dx(Sv); \quad dx(v) = -dx^*(S^{^{-1}}v), & (2.4)
\end{aligned}$$

where $\nabla^1$ (resp. $\nabla^{*1} = \nabla^3$) is the Levi-Civita connection of the first fundamental form of $x$ (resp. of $x^*$) and $\nabla^3$ denote the Levi-Civita connection of the third fundamental form $\mathrm{III}$ (recall $\mathrm{III} = \mathrm{I}^*$). For more details on the polarized hypersurface of a hypersurface immersion, see [1]. The Weingarten equations (2.4) imply the following Codazzi pairs: $(\nabla^1, S)$ and $(\nabla^3, S^{^{-1}})$. The triple $(\nabla^1, \mathrm{II}, \nabla^3)$ is conjugate, i.e. for all vector fields $u, v, w$ on $M^n$ we $w\mathrm{II}(u,v) = \mathrm{II}(\nabla^1_w u, v) + \mathrm{II}(u, \nabla^3_w v)$. $\nabla^2 = \frac{1}{2}(\nabla^1 + \nabla^3)$ is then the Levi-Civita connection of the common second fundamental form. As $\nabla^1$, $\nabla^3$ are torsion free, $C = \frac{1}{2}(\nabla^1 - \nabla^3)$ defines a symmetric $(1,2)$-tensor field. The associated cubic form $\widehat{C}(u,v,w) := \mathrm{II}(C(u,v),w)$ satisfies $2\widehat{C} = -\nabla^1\mathrm{II} = \nabla^3\mathrm{II}$ and thus is totally symmetric. The Tchebychev vector field $T$ is defined by $n\mathrm{II}(T,u) := \mathrm{tr}(v \longmapsto C(u,v))$. The traceless part $\widetilde{C}$ of the difference tensor field $C$ is given by:

$$\widetilde{C}(u,v) = C(u,v) - \frac{n}{n+2}\left(\mathrm{II}(u,T)v + \mathrm{II}(v,T)u + \mathrm{II}(u,v)T\right). \tag{2.5}$$

# 3. The equation $\widetilde{C} = 0$

Let $\bar{x}\colon \bar{M}^{n+1} \longrightarrow \mathbb{R}^{n+2}$ be a non-degenerate immersion of a connected and oriented $(n+1)$-dimensional $C^\infty$- manifold $\bar{M}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+2}$. Suppose that $\bar{x}$ lies on a quadric. There exist a constant $a \in \mathbb{R}$, a vector $b^* \in \mathbb{R}^{n+2}$ and a non-vanishing linear mapping $L\colon \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^{n+2}$, such that [6] (7.1. p 117):

$$< Lu, v >=< u, Lv >; \tag{3.6}$$
$$L\bar{x} + b^* \neq 0; \tag{3.7}$$
$$< L\bar{x} + 2b^*, \bar{x} >= a. \tag{3.8}$$

Recall the following characterization of quadrics from [6], p 117-119; note an additional remark to the proof in [3], p. 208, (2.2.b).

**3.1. Theorem.** *Let $\bar{x}\colon \bar{M}^{n+1} \longrightarrow \mathbb{R}^{n+2}$ be a non-degenerate hypersurface immersion in $\mathbb{R}^{n+2}$. Then the immersion $\bar{x}$ lies on a quadric of $\mathbb{R}^{n+2}$ if and only if $\bar{x}$ satisfies the equation $\widetilde{C} = 0$.*

Let $\bar{x}\colon \bar{M}^{n+1} \longrightarrow \mathbb{R}^{n+2}$ be a non-degenerate immersion, $L\colon \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^{n+2}$ a non-vanishing linear mapping, $b^* \in \mathbb{R}^{n+2}$ and $a \in \mathbb{R}$ such that the conditions (3.6), (3.7) and (3.8) are fulfilled. Define the function $q_{_L}\colon \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$, $y \longmapsto q_{_L}(y) =< Ly+2b^*, y >$ whose restriction to $\mathbb{S}^{n+1}(1)$ again is denoted by $q_{_L}$, and define $M^n := \{y \in \mathbb{S}^{n+1}(1)/q_{_L}(y) = a\}$.

**3.2. Proposition.** *Assume that the set $M^n$ is not empty and the gradient of $q_{_L}$ on $\mathbb{S}^{n+1}(1)$ does not vanish ($\mathrm{grad}^{\mathbb{S}^{n+1}(1)} q_{_L} \neq 0$). Then the inclusion map $M^n \hookrightarrow \mathbb{S}^{n+1}(1)$ defines a hypersurface immersion $x\colon M^n \longrightarrow \mathbb{S}^{n+1}(1)$ such that*

$$N = \frac{1}{\|\mathrm{grad}^{\mathbb{S}^{n+1}(1)} q_{_L}\|}\mathrm{grad}^{\mathbb{S}^{n+1}(1)} q_{_L} = \frac{Lx + b^* - ax+ < b^*, x > x}{\|Lx + b^* - ax+ < b^*, x > x\|} \tag{3.9}$$

*is a unit vector field normal to $M^n$.*

**Proof.** $\mathrm{grad}^{\mathbb{S}^{n+1}(1)} q_{L\mathbb{S}^{n+1}(1)} = (\mathrm{grad}^{\mathbb{R}^{n+2}} q_L)^T = (2Lx + 2b^*)^T = 2(Lx - ax + b^* + < b^*, x > x)$. The rest of the proof follows using Satz 4.5, p.167-169, in [2]. □

**3.3. Proposition.** *The unit vector fields $x$ and $N$ satisfy the following equations:*

$$< Lx, dx >= - < b^*, dx >; \qquad < Ldx, N >= Dd(\ln D), \tag{3.10}$$

*where $D = \|Lx + b^* - ax+ < b^*, x > x\|$.*

**Proof.** Straight forward calculations differentiating $0 =< x, N >$ and $1 =< N, N >$. □

**3.4. Corollary.** *The totally symmetric cubic form $\widehat{C}$ of the immersion $x$ satisfies*

$$\begin{aligned} 2\widehat{C}(u, v, w) &= \mathrm{II}(u, v)w(\ln D) + \mathrm{II}(u, w)v(\ln D) + \mathrm{II}(v, w)u(\ln D) \\ &\quad + \frac{1}{D}[< b^*, dx(u) > \mathrm{I}(v, w)+ < b^*, dx(v) > \mathrm{I}(u, w) \\ &\quad + < b^*, dx(w) > \mathrm{I}(u, v)]. \end{aligned} \tag{3.11}$$

**Proof.** The second fundamental form $\mathbb{I}$ is given by

$$\mathbb{I}(u,v) = -<dx(u), dN(v)>= -\frac{1}{D}[<dx(u), Ldx(v)> -(a- <b^*,x>)\mathrm{I}(u,v)].$$

Consequently, using the Gauß equation for $w$ and $u$ (resp. for $w$ and $v$), the equations (3.10) and the properties of the linear mapping $L$, one has:

$$\begin{aligned}
w\mathbb{I}(u,v) &= -w(\ln D)\mathbb{I}(u,v) - \frac{1}{D}\left(<\bar{\nabla}_w dx(u), Ldx(v)> + <dx(u), L\bar{\nabla}_w dx(v)>\right. \\
&\quad \left. + <b^*, dx(w)> \mathrm{I}(u,v) - (a- <b^*,x>)w\mathrm{I}(u,v)\right) \\
&= -w(\ln D)\mathbb{I}(u,v) - \frac{1}{D}\left(<dx(\nabla^1_w u) + \mathbb{I}(u,w)N - \mathrm{I}(u,w)x, Ldx(v)>\right. \\
&\quad + <Ldx(u), dx(\nabla^1_w v) + \mathbb{I}(v,w)N - \mathrm{I}(v,w)x> + <b^*, dx(w)> \mathrm{I}(u,v) \\
&\quad \left. -(a- <b^*,x>)w\mathrm{I}(u,v)\right) \\
&= -\frac{1}{D}\left(<dx(\nabla^1_w u), dx(v)> + Dv(\ln D)\mathbb{I}(u,w) + <b^*, dx(v)> \mathrm{I}(u,w)\right. \\
&\quad + <Ldx(u), dx(\nabla^1_w v)> + Du(\ln D)\mathbb{I}(v,w) + <b^*, dx(u)> \mathrm{I}(v,w) \\
&\quad \left. + <b^*, dx(w)> \mathrm{I}(u,v) - (a- <b^*,x>)w\mathrm{I}(u,v)\mathrm{I}(\nabla^1_w u, v) + \mathrm{I}(u, \nabla^1_w v)\right) \\
&\quad -w(\ln D)\mathbb{I}(u,v) \\
&= -(w(\ln D)\mathbb{I}(u,v) + u(\ln D)\mathbb{I}(v,w) + v(\ln D)\mathbb{I}(u,w)) + \mathbb{I}(\nabla^1_w u, v) + \mathbb{I}(u, \nabla^1_w v) \\
&\quad -\frac{1}{D}\left[<b^*, dx(v)> \mathrm{I}(u,w) + <b^*, dx(u)> \mathrm{I}(v,w) + <b^*, dx(w)> \mathrm{I}(u,v)\right].
\end{aligned}$$

Substitution of the foregoing expression for $w\mathbb{I}(u,v)$ into the following equation gives the assertion:

$$-2\widehat{C}(u,v,w) = (\nabla^1_w \mathbb{I})(u,v) = w\mathbb{I}(u,v) - \mathbb{I}(\nabla^1_w u, v) - \mathbb{I}(u, \nabla^1_w v).$$

□

We are going to calculate $\widetilde{C}$ in local terms. Assume that the immersion $x\colon M^n \longrightarrow \mathbb{S}^{n+1}(1)$ in Proposition 3.2 is non-degenerate. With respect to a frame $(e_i)_{1\leq i\leq n}$, one has:

$$\begin{aligned}
C^k_{ij} &= \frac{1}{2D}\left[\mathbb{I}^{kl}\mathrm{I}_{jl} <b^*, dx(e_i)> + \mathbb{I}^{kl}\mathrm{I}_{il} <b^*, dx(e_j)> + \mathbb{I}^{kl}\mathrm{I}_{ij} <b^*, dx(e_l)>\right] \\
&\quad +\frac{1}{2}\left[\mathbb{I}^{kl}\mathbb{I}_{ij} e_l \ln D + e_i \ln D \delta^k_j + e_j \ln D \delta^k_i\right].
\end{aligned}$$

($C(e_i, e_k) = C^k_{ij} e_k$, $(\mathbb{I}^{kl})$ is the inverse matrix such that $\mathbb{I}^{kl}\mathbb{I}_{lh} = \delta^k_h$.)

Particularly,

$$nT_j = \frac{n+2}{2} e_j \ln D + \frac{nH^*}{2D} <b^*, dx(e_j)> + \frac{1}{D}\mathbb{I}^{kl}\mathrm{I}_{jl} <b^*, dx(e_k)>,$$

where $H^* = \frac{1}{n}\mathrm{tr}S^{-1}$ is the mean curvature of the polarized hypersurface $x^*$. So

$$\begin{aligned}\widetilde{C}^k_{ij} &= C^k_{ij} - \frac{n}{n+2}[T_i\delta^k_j + T_j\delta^k_i + \mathrm{II}_{ij}T^k] \qquad (3.12)\\ &= \frac{1}{2D}\{\mathrm{II}^{kl}\mathrm{I}_{jl} < b^*, dx(e_i) > + \mathrm{II}^{kl}\mathrm{I}_{il} < b^*, dx(e_j) > + \mathrm{II}^{kl}\mathrm{I}_{ij} < b^*, dx(e_l) >\}\\ &\quad - \frac{1}{2(n+2)D}\{[nH^* < b^*, dx(e_i) > + 2\mathrm{II}^{sl}\mathrm{I}_{il} < b^*, dx(e_s) >]\delta^k_j\\ &\quad + [nH^* < b^*, dx(e_j) > + 2\mathrm{II}^{sl}\mathrm{I}_{jl} < b^*, dx(e_s) >]\delta^k_i\\ &\quad + \mathrm{II}^{kh}\mathrm{II}_{ij}[nH^* < b^*, dx(e_h) > + 2\mathrm{II}^{sl}\mathrm{I}_{hl} < b^*, dx(e_s) >]\}\,.\end{aligned}$$

**3.5. Theorem.** *Regular intersections of Euclidean (central) hyperquadrics, centered at the origin of a sphere $\mathbb{S}^{n+1}(1)$, with that sphere satisfy the equation $\widetilde{C} = 0$.*

**Proof.** Consequence of (3.12) with $b^* = 0$. □

Consider now the intersection of a quadric, centered at $0 \neq b^* \in \mathbb{R}^{n+2}$, with $\mathbb{S}^{n+1}(1)$.

**3.6. Lemma.** *If the constant vector $b^*$ is orthogonal to $dx(T_pM^n)$ at each point $p \in M^n$, then the immersion $x$ is totally umbilical.*

**Proof.** The assumption $< b^*, dx >= 0$ implies that $b^* \in \mathrm{Span}\{x, N\}$. There are differentiable functions $\alpha$ and $\beta$ such that $b^* = \beta x + \alpha N$. Since $b^*$ is constant one has

$$0 = (d\beta)x + \beta dx + (d\alpha)N + \alpha dN.$$

By the linear independence of $dx$, $x$ and $N$, we get $\beta dx = -\alpha dN$ and $d\beta = 0 = d\alpha$. □

**3.7. Theorem.** *A regular intersection of $\mathbb{S}^{n+1}(1)$ with a quadric of $\mathbb{R}^{n+2}$, not centered at the origin, satisfies the equation $\widetilde{C} = 0$ if and only if it is umbilical.*

**Proof.** Assume that the immersion satisfies $\widetilde{C} = 0$. Choose a frame $(e_i)_{1\leq i\leq n}$ of principal vectors ($Se_i = \lambda_i e_i$, $\lambda_i$ is a principal curvature function). For any fixed $1 \leq i \leq n$ and any fixed pair $1 \leq k \neq i \leq n$, the equation (3.12) implies:

$$0 = \widetilde{C}^i_{ii} = \frac{3n}{2(n+2)D}(\lambda_i^{-1} - H^*) < b^*, dx(e_i) >, \qquad (3.13)$$

$$0 = \widetilde{C}^k_{ik} = \frac{< b^*, dx(e_i) >}{2(n+2)D}\left((n+2)\lambda_k^{-1} - nH^* - 2\lambda_i^{-1}\right). \qquad (3.14)$$

If $< b^*, dx >= 0$, then from Lemma 3.6, the immersion is umbilical. Suppose now that $< b^*, dx(e_i) > \neq 0$ for some $1 \leq i \leq n$. From (3.13) we have $\lambda_i^{-1} = H^*$; and then for any $1 \leq k (\neq i) \leq n$ the equation (3.14) implies $\lambda_k^{-1} = H^*$. Thus $x^*$ (and then also $x$) is umbilical. □

**3.8. Remark.**

(i) Let $x\colon M^n \longrightarrow \mathbb{S}^{n+1}(1)$ be an immersion of a regular hypersurface $M^n$ into $\mathbb{S}^{n+1}(1)$ satisfying the equation $\widetilde{C} = 0$. From Theorem 3.7, if $x$ is non-isoparametric and $x(M^n)$ is contained in a quadric $\mathbb{Q}^{n+1} \neq \mathbb{S}^{n+1}(1)$ of $\mathbb{R}^{n+2}$, then $\mathbb{Q}^{n+1}$ is centered at the origin. We use this to construct quadrics containing regular non-isoparametric surfaces fulfilling the condition $\widetilde{C} = 0$.

(ii) In the relative differential geometry of hypersurfaces the traceless tensor $\widetilde{C}$ (see the introduction) is invariant under a change of normalization. Thus, to prove that the equation $\widetilde{C} = 0$ implies that the immersion lies on a quadric, one can restrict to the equiaffine normalization [6]. For immersions in spheres, a similar proof is not possible. For the special case of surfaces ($n = 2$) in $\mathbb{S}^3(1)$, we have the following situation:

**3.9. Theorem ([4]).** *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a regular surface immersion of a connected and orientable 2-dimensional $C^\infty$-manifold $M^2$ into $\mathbb{S}^3(1)$ without umbilics. The immersion $x$ satifies the equation $\widetilde{C} = 0$ if and only if either (locally) $x$ is isoparametric or there exist an open interval $I$ of constant sign, constants $k_1, k_2, k_3 \in \mathbb{R}$ and constant vectors $C_1, C_2, C_3, C_4 \in \mathbb{R}^4$ such that the constants satisfy* (i)-(v) *and $x$ is given by* (vi)*:*

(i) $k_2 \neq 0 \neq k_1, \qquad k_3 < 1; \qquad 1 - 2k_3 - k_1 > 0, \qquad k_3^2 + k_1 > 0;$

(ii) $1 - k_1u^2 > 0, \qquad k_1u^4 - 2k_3u^2 - 1 > 0,$ *for all $u \in I$;*

(iii) *$C_1$ and $C_2$ are orthonormal;*

(iv) *$C_1$ and $C_2$ are orthogonal to $C_3$ and $C_4$;*

(v) $< C_3, C_3 >= \frac{1-k_3}{1-2k_3-k_1} =< C_4, C_4 >; \quad < C_3, C_4 >= \frac{\sqrt{k_3^2+k_1}}{1-2k_3-k_1};$

(vi) *$x$ is represented by the following surface in $\mathbb{S}^3(1) \subset \mathbb{R}^4$:*

$$\mathbb{L} = \{\overset{-1}{\rho}\,\overset{-1}{k_4}\,(C_1 \cos k_4 v + C_2 \sin k_4 v) + C_3 \cos \frac{1}{2}\gamma(u) + C_4 \sin \frac{1}{2}\gamma(u) : (u, v) \in I \times \mathbb{R}\},$$

*where* $\overset{-2}{k_4} = k_2^2(1 - 2k_3 - k_1); \quad \overset{-2}{k_4}\,\overset{-2}{\rho}(u) = \frac{k_1+k_3}{2k_3+k_1-1} + \frac{\sqrt{k_1+k_3^2}}{1-2k_3-k_1} \sin \gamma(u)$ *and* $\gamma(u) = \arctan(\frac{1+k_3u^2}{\sqrt{k_1u^4-2k_3u^2-1}})$.

**3.10. Theorem.** *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a regular surface immersion of a connected and orientable 2-dimensional $C^\infty$-manifold $M^2$ into $\mathbb{S}^3(1)$. Assume that the immersion $x$ is non-isoparametric and has no umbilics. Then the immersion $x$ satifies the condition $\widetilde{C} = 0$ if and only if there exists a central quadric $\mathbb{Q}^3 \neq \mathbb{S}^3(1)$ of $\mathbb{R}^4$ centered at the origin such that $x(M^2) \subset \mathbb{Q}^3$.*

**Proof.** The first part of the proof follows from Theorem 3.5. For the second part we apply Theorem 3.9, assuming $\widetilde{C} = 0$. Under our additional assumptions, there exist an open interval $I$ of constant sign, constants $k_1, k_2, k_3 \in \mathbb{R}$ and constant vectors $C_1, C_2, C_3, C_4 \in \mathbb{R}^4$

satisfying the properties (i), (ii) (iii), (iv), (v) and (vi) in Theorem 3.9. It is clear that the vectors $C_1, C_2, C_3, C_4$ constitute a basis of $\mathbb{R}^4$, $C_1, C_2 \in \mathbb{S}^3(1)$ and $C_3, C_4 \notin \mathbb{S}^3(1)$. Since the immersion $x$ is not umbilical from Remark 3.8 (i), one has to choose $b^* = 0$. Let $L\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a linear mapping satisfying (3.6) with $b^* = 0$. One has:

$$\begin{aligned}
< Lx, x > &= \rho^{-2} k_4^{-2} (< LC_1, C_1 > \cos^2 k_4 v + < LC_2, C_2 > \sin^2 k_4 v + < LC_1, C_2 > \sin 2k_4 v) \\
&\quad + < LC_3, C_3 > \cos^2 \frac{\gamma(u)}{2} + < LC_4, C_4 > \sin^2 \frac{\gamma(u)}{2} + < LC_3, C_4 > \sin \frac{\gamma(u)}{2} \\
&\quad + 2\rho^{-1} k_4^{-1} \left( < LC_1, C_3 > \cos k_4 v \cos \frac{\gamma(u)}{2} + < LC_1, C_4 > \cos k_4 v \sin \frac{\gamma(u)}{2} \right. \\
&\quad \left. + < LC_2, C_3 > \sin k_4 v \cos \frac{\gamma(u)}{2} + < LC_2, C_4 > \sin k_4 v \sin \frac{\gamma(u)}{2} \right) \\
&= a = const.
\end{aligned}$$

By the linear independence of the functions 1, $\sin k_4 v$, $\cos k_4 v$, $\sin 2k_4 v$ and $\cos 2k_4 v$, we get the following equations:

$$\begin{aligned}
a &= \frac{1}{2}\rho^{-2} k_4^{-2} (< LC_1, C_1 > + < LC_2, C_2 >) + < LC_3, C_3 > \cos^2 \frac{\gamma(u)}{2} && (3.15) \\
&\quad + < LC_4, C_4 > \sin^2 \frac{\gamma(u)}{2} + < LC_3, C_4 > \sin \gamma(u); \\
0 &= \frac{1}{2}\rho^{-2} k_4^{-2} (< LC_1, C_1 > - < LC_2, C_2 >); && (3.16) \\
0 &= \rho^{-2} k_4^{-2} < LC_1, C_2 >; && (3.17) \\
0 &= 2\rho^{-1} k_4^{-1} \left( < LC_1, C_3 > \cos \frac{\gamma(u)}{2} + < LC_1, C_4 > \sin \frac{\gamma(u)}{2} \right); && (3.18) \\
0 &= 2\rho^{-1} k_4^{-1} \left( < LC_2, C_3 > \cos \frac{\gamma(u)}{2} + < LC_2, C_4 > \sin \frac{\gamma(u)}{2} \right). && (3.19)
\end{aligned}$$

From the equations (3.17), (3.18), (3.19), and the linear independence of the functions $\cos \frac{\gamma(u)}{2}$ and $\sin \frac{\gamma(u)}{2}$, one has:

$$\begin{aligned}
&< LC_1, C_2 >= 0, \quad < LC_1, C_3 >= 0, \quad < LC_1, C_4 >= 0; \\
&< LC_2, C_1 >= 0, \quad < LC_2, C_3 >= 0, \quad < LC_2, C_4 >= 0.
\end{aligned}$$

So $LC_1 \in \mathrm{Span}\{C_1\}$, $LC_2 \in \mathrm{Span}\{C_2\}$ and $LC_3, LC_4 \in \mathrm{Span}\{C_3, C_4\}$. The equation (3.16) gives $< LC_1, C_1 >=< LC_2, C_2 >$. Rewriting (3.15), one has:

$$\begin{aligned}
a &= \frac{(k_1 + k_3) < LC_1, C_1 >}{2k_3 + k_1 - 1} + \left( \frac{\sqrt{k_1 + k_3^2} < LC_1, C_1 >}{1 - 2k_3 - k_1} + < LC_3, C_4 > \right) \sin \gamma(u) && (3.20) \\
&\quad + \frac{1}{2}(< LC_3, C_3 > + < LC_4, C_4 >) + \frac{1}{2}(< LC_3, C_3 > - < LC_4, C_4 >) \cos \gamma(u).
\end{aligned}$$

By the linear independence of the functions 1, $\sin\gamma(u)$ and $\cos\gamma(u)$, we get:

$$a = \frac{1}{2}(< LC_3, C_3 > + < LC_4, C_4 >) + \frac{(k_1+k_3) < LC_1, C_1 >}{2k_3+k_1-1}; \tag{3.21}$$

$$0 = \frac{1}{2}(< LC_3, C_3 > - < LC_4, C_4 >); \tag{3.22}$$

$$0 = \frac{\sqrt{k_1+k_3^2} < LC_1, C_1 >}{1-2k_3-k_1} + < LC_3, C_4 > . \tag{3.23}$$

The equation (3.22) implies $< LC_3, C_3 >=< LC_4, C_4 >$, and then from (3.21)

$$< LC_3, C_3 >=< LC_4, C_4 > = a - \frac{(k_1+k_3) < LC_1, C_1 >}{2k_3+k_1-1}. \tag{3.24}$$

Therefore the linear mapping $L$ has to be chosen such that

$$LC_3 = a_1C_3 + a_2C_4 \tag{3.25}$$

$$LC_4 = a_2C_3 + a_1C_4, \tag{3.26}$$

where

$$a_1 = \bar{a}(1-k_3) - \bar{b}\sqrt{k_1+k_3^2}, \qquad a_2 = \bar{b}(1-k_3) - \bar{a}\sqrt{k_1+k_3^2} \quad \text{and}$$

$$\bar{a} = a - \frac{(k_1+k_3) < LC_1, C_1 >}{2k_3+k_1-1}, \qquad \bar{b} = -\frac{\sqrt{k_1+k_3^2} < LC_1, C_1 >}{1-2k_3-k_1}.$$

Since $k_1 + k_3^2 \neq 0$, the vector $LC_1$ can not be zero. So one has to choose $< LC_1, C_1 >\neq 0$. Choose $a \neq< LC_1, C_1 >\in \mathbb{R}$ such that $\bar{a} = a - \frac{(k_1+k_3)<LC_1,C_1>}{2k_3+k_1-1} \neq 0$. Since the vectors $C_1, C_2, C_3, C_4$ are linearly independent, the mapping $L$ and the immersion $x$ satisfy (3.7), i.e. $Lx \neq 0$; otherwise

$$< LC_1, C_1 > \cos k_4 v = 0, \quad < LC_1, C_1 > \sin k_4 v = 0, \tag{3.27}$$

$$a_1 \cos\frac{\gamma(u)}{2} + a_2 \sin\frac{\gamma(u)}{2} = 0, \tag{3.28}$$

$$a_1 \sin\frac{\gamma(u)}{2} + a_2 \cos\frac{\gamma(u)}{2} = 0; \tag{3.29}$$

the equations in (3.27) imply $< LC_1, C_1 >= 0$, and from the equations (3.28) and (3.29) one has $a_1 = 0 = a_2$, and then $\bar{a} = 0$ (a contradiction). Thus the linear mapping $L$, satisfying the properties above, defines a quadric $\mathbb{Q}^3 \neq \mathbb{S}^3(1)$ of $\mathbb{R}^4$ which contains the immersion $x$. $\square$

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